A fast and high-order HDG solver for the high-frequency Helmholtz equation

Matthias Taus

Postdoctoral Associate,
Department of Mathematics

In collaboration with Leonardo Zepeda-Núñez and Laurent Demanet

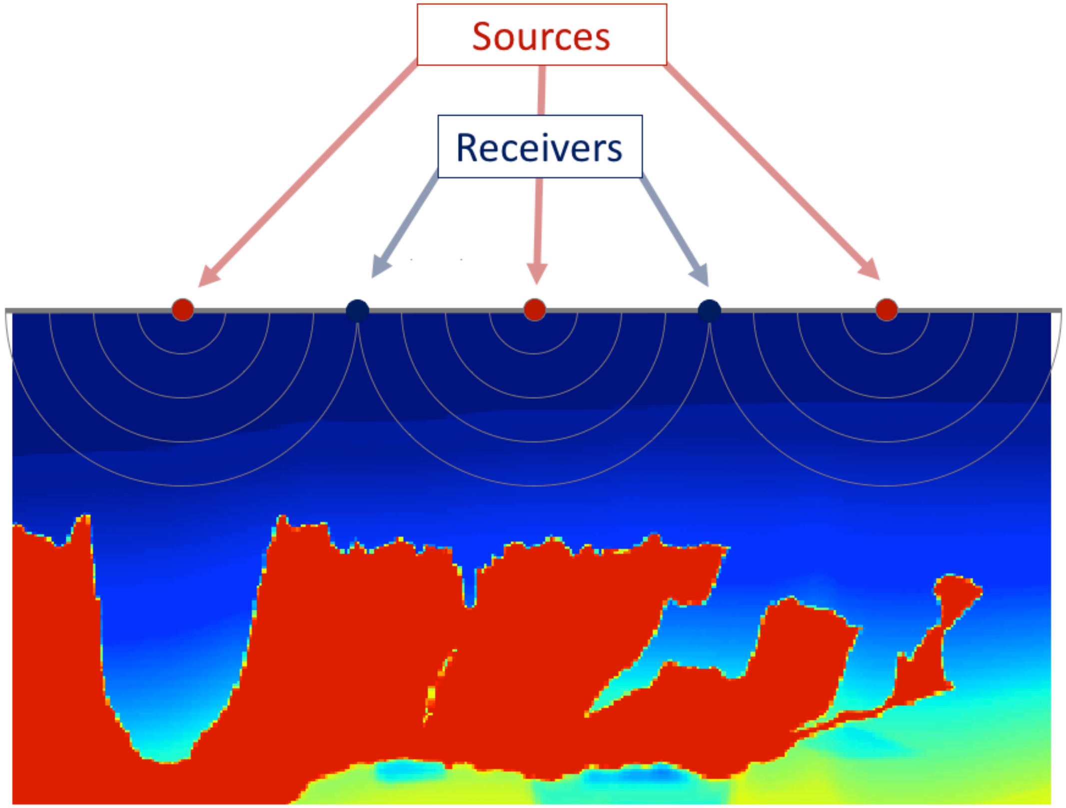
MIT Earth Resources Laboratory
2016 Annual Founding Members Meeting
May 19, 2016

Earth Resources Laboratory

Massachusetts Institute of Technology

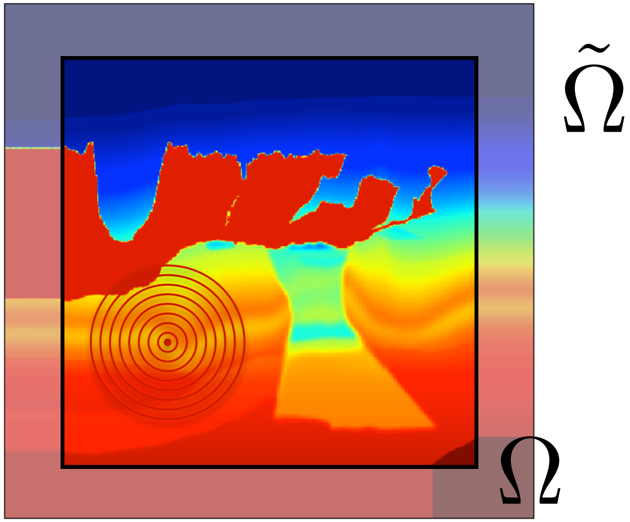
Motivation

Wave propagation in inhomogeneous media



Model Problem

$$-\Delta u - \omega^2 m u = f \quad \text{in } \Omega$$
$$+\text{A.B.C on } \partial\Omega$$



Absorbing boundary conditions: Perfectly Matched Layers (PMLs)

$$-\operatorname{div}\left(\tilde{\Lambda}\nabla u\right) - \omega^2 \tilde{m} u = \tilde{f} \quad \text{in } \tilde{\Omega}$$

$$u = 0 \quad \text{on } \partial\tilde{\Omega}$$

Existing discretization techniques

1. Finite Difference Methods (FD)
2. Integral Equation Methods (IE)
3. Finite Element Methods (FE)

	FD	IE	FE
Inhomogeneous media	✔	✖	✔
Flexible meshes	✖	✔	✔
Non-smooth approximations	✖	✔	✔
High-order approximations	✔	✔	✔

Discretization

High-frequency problems → High-order approximations

Inhomogeneous wave speeds → Non-smooth approximations

→ Discontinuous Galerkin Finite Element Methods (DGFEMs)

Efficiency

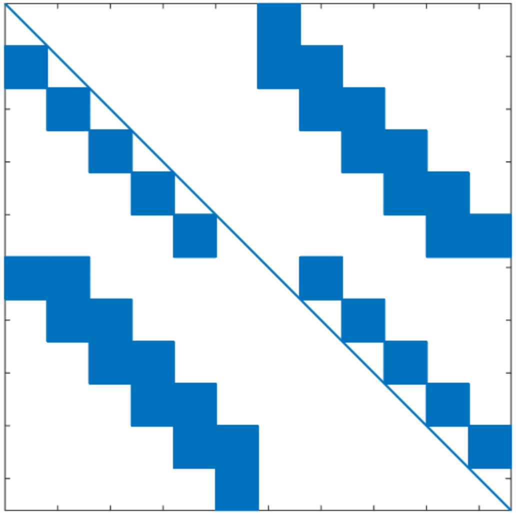
- Assembly of the system matrix: $O(N)$
- Solution of the resulting linear system:
  - Direct solver: $O(N^3)$
  - Iterative solver: $O(\omega N^2)$
  - Preconditioned iterative solver: $O(\omega N)$

Problem:
Even preconditioned iterative solvers are not feasible for high-frequency problems!

Solution: Method of Polarized Traces

The Method of Polarized Traces (L. Zepeda-Núñez)

1. Divide the domain into layers
2. Reduce the problem to the degrees of freedom λ on the interfaces by local (direct) inversions
3. Split $\lambda = \lambda^{\uparrow} + \lambda^{\downarrow}$ to get a well-structured matrix:



4. Precondition with the upper- and lower-triangular matrix.

Existing applications

Low-order methods (FD and standard FE):

\+ Solution in $O(N)$ time, independently of frequency ω

\+ Parallelism

Problem:

For high-order methods, the problem has to be reduced to a layer of unknowns around each interface.

Solution: Discontinuous basis functions and DGFEMs

Review

DGFEMs + Method of Polarized Traces

Accuracy:

+ Non-smooth approximations
+ High-order approximations
+ Flexible Meshes

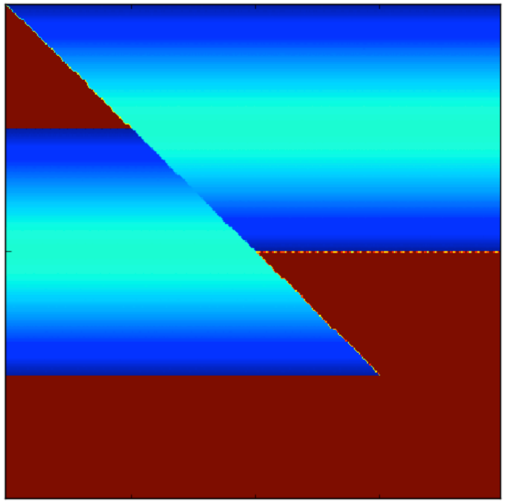
Efficiency:

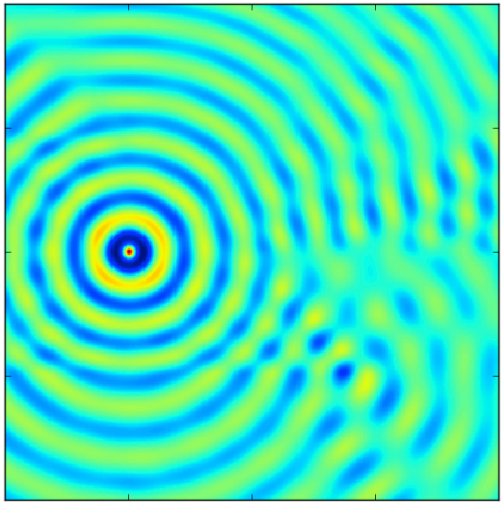
+ $O(N)$ time
+ Parallelism
+ Efficiency independent of the order

Additional desirable features:

+ Adaptively refined meshes
+ Arbitrary geometries

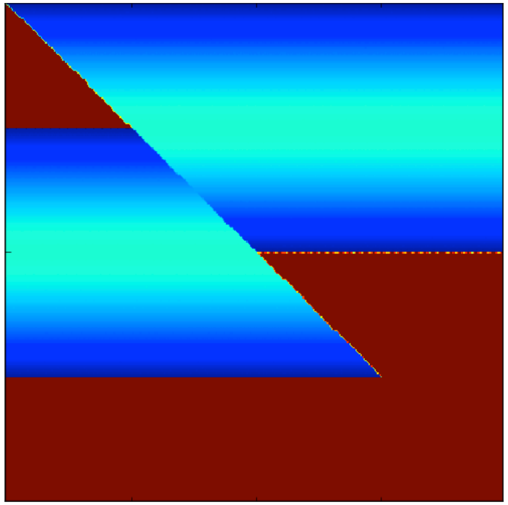
Example 1: Accuracy

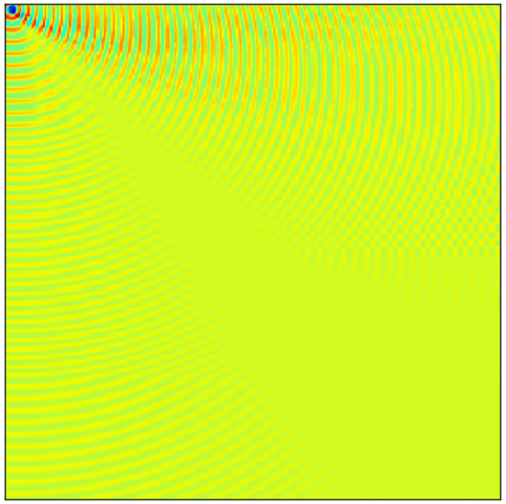




h	$p=1$ Error	eoc	$p=2$ Error	eoc	$p=3$ Error	eoc
1.4E-1	110%		15%		1.9%	
7.1E-2	75%	0.6	1.4%	3.4	0.1%	4.2
3.5E-2	22%	1.7	0.11%	3.7	$7 \cdot 10^{-3}$%	3.9
1.8E-2	5.8%	1.9	10^{-3}%	3.4	$4.7 \cdot 10^{-4}$%	3.9

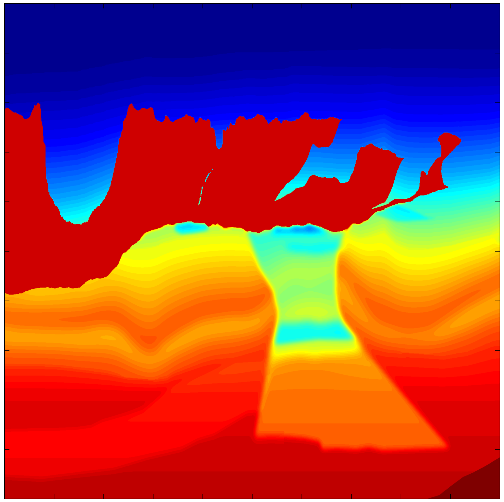
Example 1: Efficiency

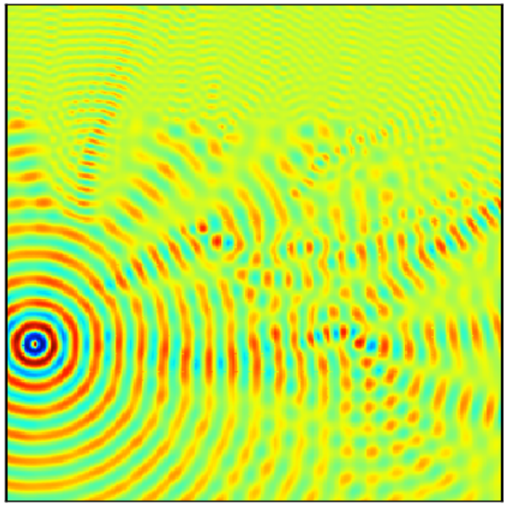




h	$\frac{\omega}{2\pi}$	L	Avg. number of iterations	Avg. time per iteration	Avg. total time
2.4E-2	5.21	3	3.5	1.43s	4.95s
1.2E-2	10.42	7	2.5	10.16s	25.19s
6.0E-3	20.84	15	2.4	47.73s	114.56s

Example 2





h	$\frac{\omega}{2\pi}$	$L=3$	$L=7$	$L=15$
1.20E-2	7.38	2.86 (**2.8**)	8.72 (**3**)	19.21 (**3**)
5.99E-3	14.75	7.21 (**3.5**)	18.76 (**3.8**)	38.25 (**4**)
3.00E-3	29.50	22.37 (**4.4**)	52.30 (**4.9**)	86.57 (**5**)

Future Work

- Extension to 3D
- Spectral discretizations
- Adaptive refinements