Earth
Resources
Laboratory

Lift and Relax for waveform inversion

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[Tarantola, 1984] [J. Virieux and S. Operto, 2009]

Motivation

Full waveform inversion

wavefield data

$$\min_{\mathbf{m},\mathbf{u}} f_{\mathrm{f}}(\mathbf{m},\mathbf{u}) = \frac{1}{2}\sum_{i,j}^{n_{\mathrm{s}},n_{\mathrm{f}}} \|\mathbf{P}\mathbf{u}_{i,j} - \mathbf{d}_{i,j}\|_2^2,$$

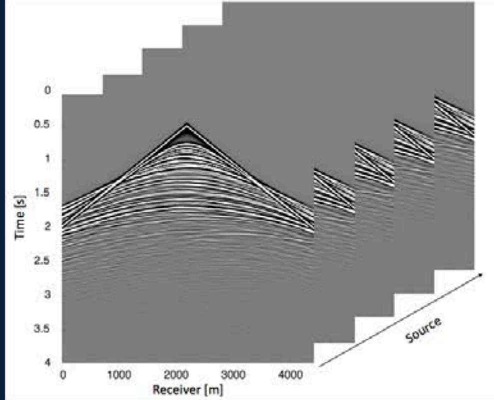
$$\text{subject to} \quad (\Delta + \omega_j^2\mathbf{m})\mathbf{u}_{i,j} = \mathbf{q}_{i,j}$$

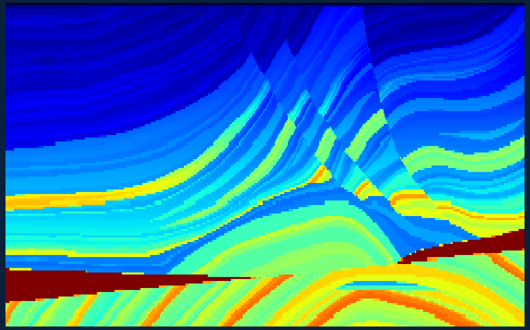
frequency

squared slowness

source

data





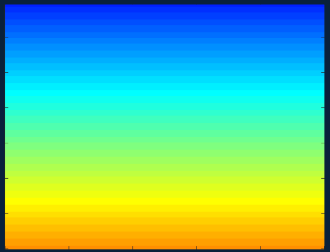
model

Motivation

Reduced approach

$$\min_{\mathbf{m}} f_{\mathrm{r}}(\mathbf{m}) = \frac{1}{2} \sum_{i,j}^{n_{\mathrm{s}},n_{\mathrm{f}}} \|\mathbf{P}\mathbf{A}_j(\mathbf{m})^{-1}\mathbf{q}_{i,j} - \mathbf{d}_{i,j}\|_2^2,$$

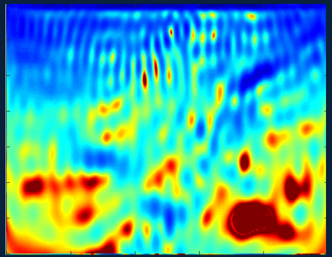
$$\text{with } \mathbf{A}_j(\mathbf{m}) = \Delta + \omega_j^2 \mathbf{m}$$



Lack of low frequency data

Lack of far offset data

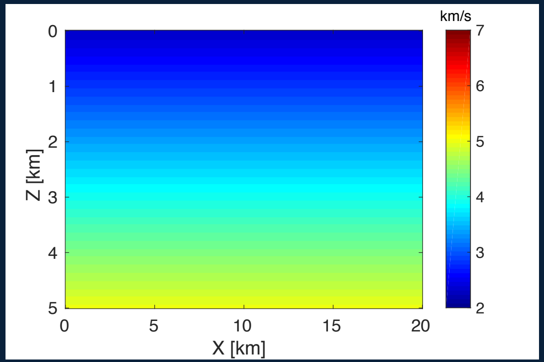
Lack of a good initial model



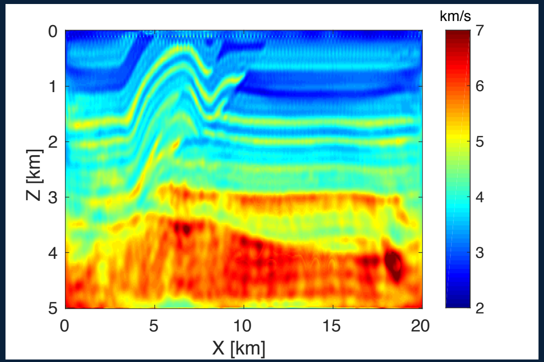
Poor inversion result

Motivation

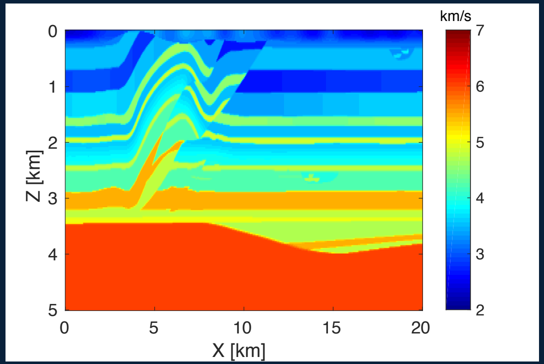
Goal: mitigate the local minima



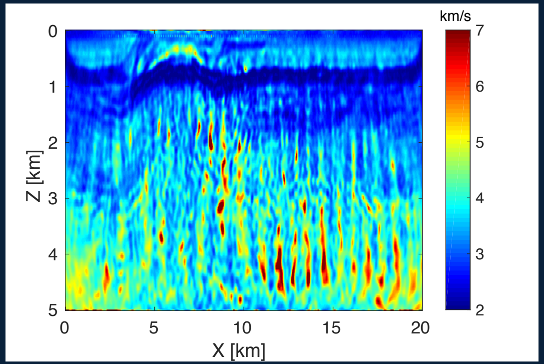
(a) Initial model



(c) Result of LRWI



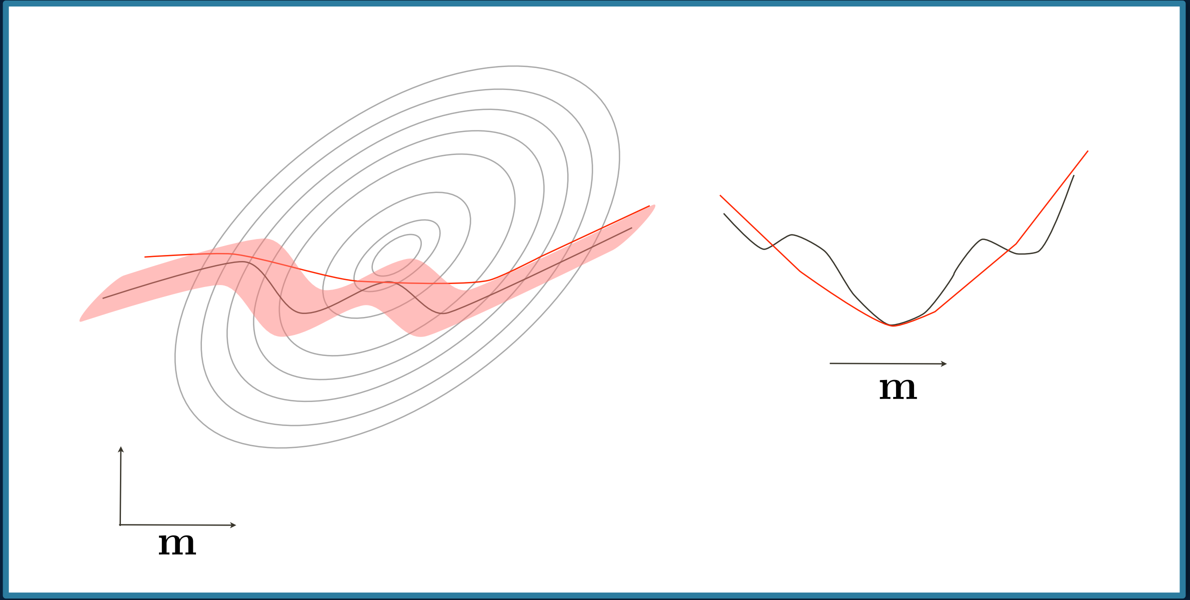
(b) True model



(d) Result of FWI

Motivation

Expand the search space



Approaches to expand the search space:

1. Relax the wave-equation constraint [van Leeuwen *et al* 2013]

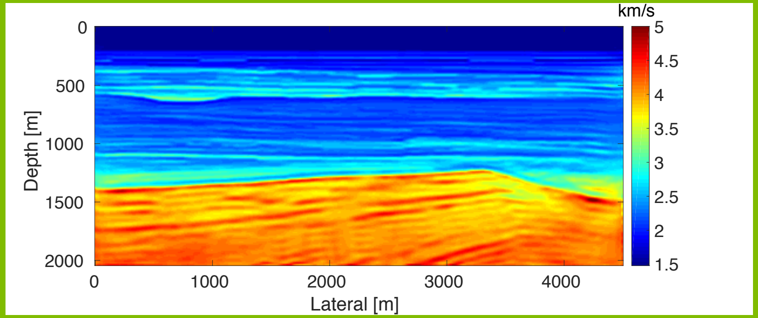
2. Lift the unknown parameters [Cosse, Demanet 2015]

Motivation

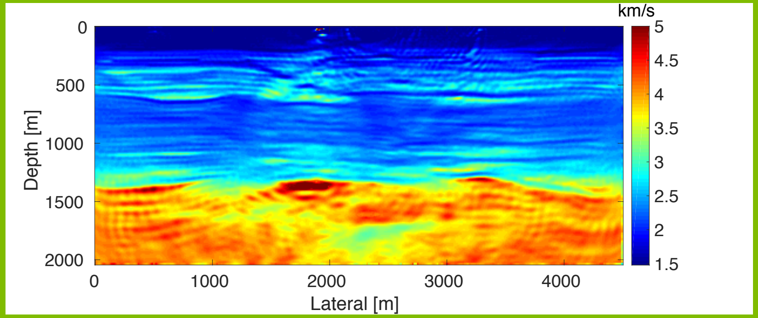
(a) Relax wave equation constraint

(wavefield reconstruction inversion (WRI))

$$\min_{\mathbf{m},\mathbf{u}} f_{\mathrm{p}}(\mathbf{m},\mathbf{u}) = \frac{1}{2}\|\mathbf{P}\mathbf{u}-\mathbf{d}\|_2^2 + \frac{\lambda}{2}\|(\Delta+\omega^2\mathbf{m})\mathbf{u}-\mathbf{q}\|_2^2.$$



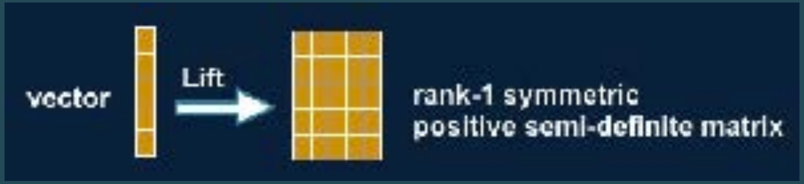
(a) WRI result

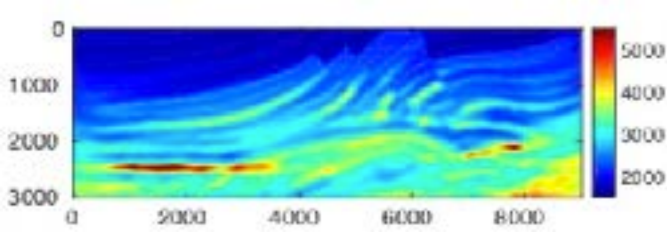


(b) FWI result

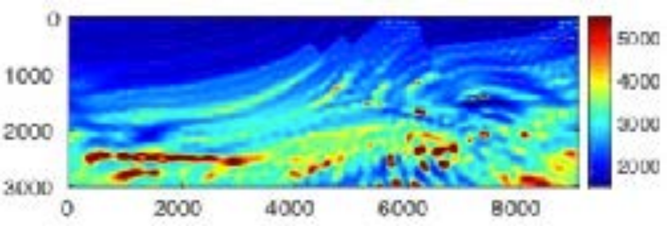
(b) Lift unknown parameters

$$[1,\mathbf{m}^\top,\mathbf{u}^\top]^\top[1,\mathbf{m},\mathbf{u}] = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \mathbf{X}_{13} \\ \mathbf{X}_{21} & \mathbf{X}_{22} & \mathbf{X}_{23} \\ \mathbf{X}_{31} & \mathbf{X}_{32} & \mathbf{X}_{33} \end{bmatrix} = \mathbf{X}$$





(c) Lifting result



(d) FWI result

Lift and Relax Waveform Inversion (LRWI)

LRWI = WRI + Lift

LRWI

New objective function

$$\begin{aligned}\min_{\mathbf{X}} f_{\mathrm{px}}(\mathbf{X}) = \frac{1}{2}\|\mathbf{P}\mathbf{X}_{31} - \mathbf{d}\|_2^2 + \frac{\lambda}{2}\|\Delta\mathbf{X}_{31} + \omega^2\mathrm{diag}(\mathbf{X}_{32}) - \mathbf{q}\|_2^2,\\ \text{subject to} \quad \mathbf{X}_{11} = 1,\\ \mathbf{X} \succeq 0,\\ \mathrm{rank}(\mathbf{X}) = 1\end{aligned}$$

with

$$\mathbf{X}_{31} = \mathbf{u}$$

$$\mathrm{diag}(\mathbf{X}_{32}) = \mathbf{m} \odot \mathbf{u}$$

Search space is too large

LRWI

Rank-2 relaxation (*r* = 2)

$$\mathbf{m} = \alpha_1 \tilde{\mathbf{m}}_1 + \alpha_2 \tilde{\mathbf{m}}_2,$$

$$\mathbf{u} = \alpha_1 \tilde{\mathbf{u}}_1 + \alpha_2 \tilde{\mathbf{u}}_2,$$

$$\mathbf{m} \odot \mathbf{u} = \tilde{\mathbf{m}}_1 \odot \tilde{\mathbf{u}}_1 + \tilde{\mathbf{m}}_2 \odot \tilde{\mathbf{u}}_2,$$

$$1 = \alpha_1^2 + \alpha_2^2$$

Rank-2 objective function

$$\min_{\tilde{\mathbf{m}},\tilde{\mathbf{u}},\alpha} f_{\mathrm{p}_2}(\tilde{\mathbf{m}},\tilde{\mathbf{u}},\alpha) = \frac{1}{2}\|\sum^{2} \mathbf{P}\alpha_l \tilde{\mathbf{u}}_l - \mathbf{d}\|_2^2 + \frac{\lambda}{2}\|\sum_{l=1}^{2} \alpha_l \Delta \tilde{\mathbf{u}}_l + \omega^2 \sum_{l=1}^{2} \tilde{\mathbf{m}}_l \odot \tilde{\mathbf{u}}_l - \mathbf{q}\|_2^2$$

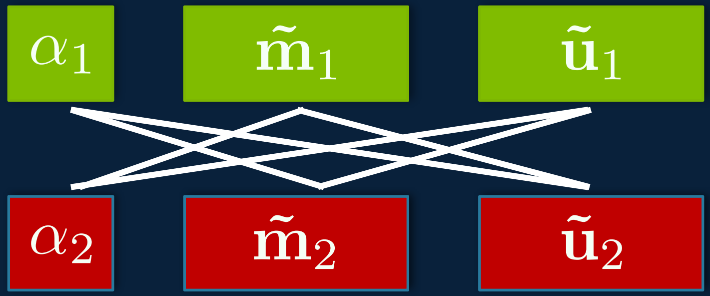
LRWI

Rank-2 objective function w/ rank-1 penalty

$$\min_{\tilde{\mathbf{m}},\tilde{\mathbf{u}},\alpha} f_{\mathrm{p}_2}(\tilde{\mathbf{m}},\tilde{\mathbf{u}},\alpha) = \frac{1}{2}\|\sum_{l=1}^{2}\mathbf{P}\alpha_l\tilde{\mathbf{u}}_l - \mathbf{d}\|_2^2 + \frac{\lambda}{2}\|\sum_{l=1}^{2}\alpha_l\Delta\tilde{\mathbf{u}}_l + \omega^2\sum_{l=1}^{2}\tilde{\mathbf{m}}_l\odot\tilde{\mathbf{u}}_l - \mathbf{q}\|_2^2$$
$$+\frac{\gamma}{2}\|\tilde{\mathbf{m}}_1\odot\tilde{\mathbf{u}}_2 - \tilde{\mathbf{m}}_2\odot\tilde{\mathbf{u}}_1\|_2^2,$$
$$\text{subject to}\quad \alpha_1^2+\alpha_2^2 = 1$$

Options for rank-1 penalties

$$\alpha_1\tilde{\mathbf{m}}_2 = \alpha_2\tilde{\mathbf{m}}_1,$$
$$\alpha_1\tilde{\mathbf{u}}_2 = \alpha_2\tilde{\mathbf{u}}_1,$$
$$\tilde{\mathbf{m}}_1\odot\tilde{\mathbf{u}}_2 = \tilde{\mathbf{m}}_2\odot\tilde{\mathbf{u}}_1$$



LRWI

Rank-2 objective function w/o constraints

$$\min_{\tilde{\mathbf{m}},\tilde{\mathbf{u}},\theta} f_{\mathrm{p}_2}(\tilde{\mathbf{m}},\tilde{\mathbf{u}},\theta) = \frac{1}{2}\|\mathbf{P}(\sin\theta\tilde{\mathbf{u}}_1 + \cos\theta\tilde{\mathbf{u}}_2) - \mathbf{d}\|_2^2$$
$$+\frac{\lambda}{2}\|\Delta(\sin\theta\tilde{\mathbf{u}}_1 + \cos\theta\tilde{\mathbf{u}}_2) + \omega^2\sum_{l=1}^{2}\tilde{\mathbf{m}}_l\odot\tilde{\mathbf{u}}_l - \mathbf{q}\|_2^2$$
$$+\frac{\gamma}{2}\|\tilde{\mathbf{m}}_1\odot\tilde{\mathbf{u}}_2 - \tilde{\mathbf{m}}_2\odot\tilde{\mathbf{u}}_1\|_2^2$$

with

$$\alpha_1 = \sin\theta \quad \text{and} \quad \alpha_2 = \cos\theta$$

LRWI

Optimization algorithm

For fixed $\tilde{\mathrm{m}}$ and θ, there is an $\tilde{\mathbf{u}}^{\star}$ satisfying $\nabla_{\tilde{\mathbf{u}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}} = 0$ given by the following expression:

$$\tilde{\mathbf{u}}^{\star} = (\tilde{\mathbf{S}}^{\top}\tilde{\mathbf{S}})^{-1}\tilde{\mathbf{S}}^{\top}\begin{bmatrix}\mathbf{d}\\ \lambda^{\frac{1}{2}}\mathbf{q}\\ 0\end{bmatrix}$$

where

$$\tilde{\mathbf{S}} = \begin{bmatrix}\sin\theta\mathbf{P} & \cos\theta\mathbf{P}\\ \lambda^{\frac{1}{2}}\tilde{\mathbf{A}}(\tilde{\mathbf{m}}_1) & \lambda^{\frac{1}{2}}\tilde{\mathbf{A}}(\tilde{\mathbf{m}}_2)\\ \gamma^{\frac{1}{2}}\mathrm{diag}(\tilde{\mathbf{m}}_2) & -\gamma^{\frac{1}{2}}\mathrm{diag}(\tilde{\mathbf{m}}_1)\end{bmatrix},$$

$$\text{with}\quad \tilde{\mathbf{A}}(\tilde{\mathbf{m}}_1) = \lambda^{\frac{1}{2}}(\sin\theta\Delta + \omega^2\tilde{\mathbf{m}}_1), \quad \text{and} \quad \tilde{\mathbf{A}}(\tilde{\mathbf{m}}_2) = \lambda^{\frac{1}{2}}(\cos\theta\Delta + \omega^2\tilde{\mathbf{m}}_2).$$

LRWI

Gradients with respect to $\tilde{\mathrm{m}}$ and θ :

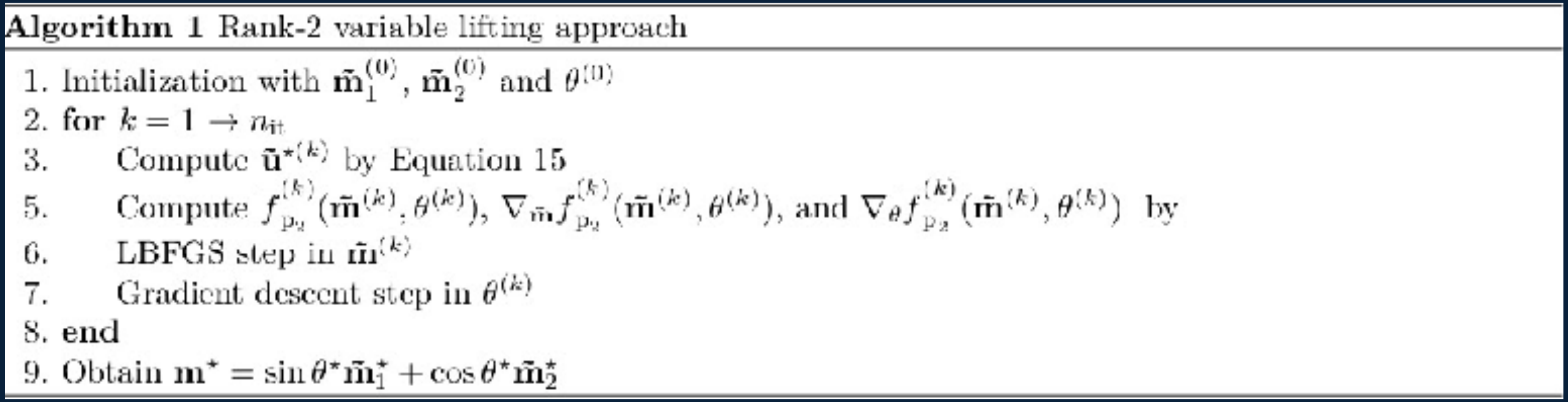
$$\begin{aligned}
\nabla_{\tilde{\mathbf{m}}} \overline{f}_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \theta) &= \nabla_{\tilde{\mathbf{m}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}^{\star}(\tilde{\mathbf{m}}, \theta), \theta) = \nabla_{\tilde{\mathbf{m}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}} + \cancel{\nabla_{\tilde{\mathbf{u}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}} \nabla_{\tilde{\mathbf{m}}} \tilde{\mathbf{u}}}, \\
\nabla_{\theta} \overline{f}_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \theta) &= \nabla_{\theta} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}^{\star}(\tilde{\mathbf{m}}, \theta), \theta) = \nabla_{\theta} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}} + \cancel{\nabla_{\tilde{\mathbf{u}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}} \nabla_{\theta} \tilde{\mathbf{u}}}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\nabla_{\tilde{\mathbf{m}}} \overline{f}_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \theta) &= \nabla_{\tilde{\mathbf{m}}} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}}, \\
\nabla_{\theta} \overline{f}_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \theta) &= \nabla_{\theta} f_{\mathrm{p}_2}(\tilde{\mathbf{m}}, \tilde{\mathbf{u}}, \alpha)|_{\tilde{\mathbf{u}}=\tilde{\mathbf{u}}^{\star}}.
\end{aligned}$$

LRWI

Optimization algorithm



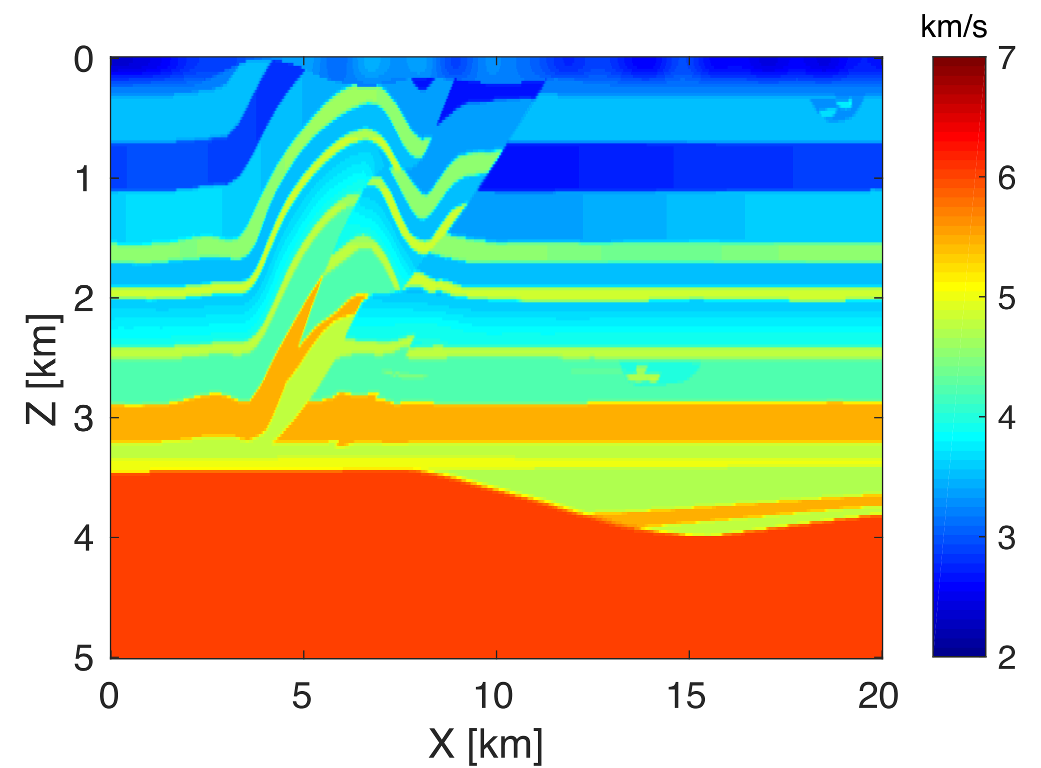
Algorithm 1 Rank-2 variable lifting approach

```
Initialization with $\tilde{\mathbf{m}}_1^{(0)}$, $\tilde{\mathbf{m}}_2^{(0)}$ and $\theta^{(0)}$
for $k = 1 \rightarrow n_{\mathrm{it}}$
    Compute $\tilde{\mathbf{u}}^{\star(k)}$ by Equation 15
    Compute $f_{\mathrm{p}_2}^{(k)}(\tilde{\mathbf{m}}^{(k)}, \theta^{(k)})$, $\nabla_{\tilde{\mathbf{m}}} f_{\mathrm{p}_2}^{(k)}(\tilde{\mathbf{m}}^{(k)}, \theta^{(k)})$, and $\nabla_{\theta} f_{\mathrm{p}_2}^{(k)}(\tilde{\mathbf{m}}^{(k)}, \theta^{(k)})$ by
    LBFGS step in $\tilde{\mathbf{m}}^{(k)}$
    Gradient descent step in $\theta^{(k)}$
end
Obtain $\mathbf{m}^{\star} = \sin\theta^{\star}\tilde{\mathbf{m}}_1^{\star} + \cos\theta^{\star}\tilde{\mathbf{m}}_2^{\star}$
```

Numerical experiment

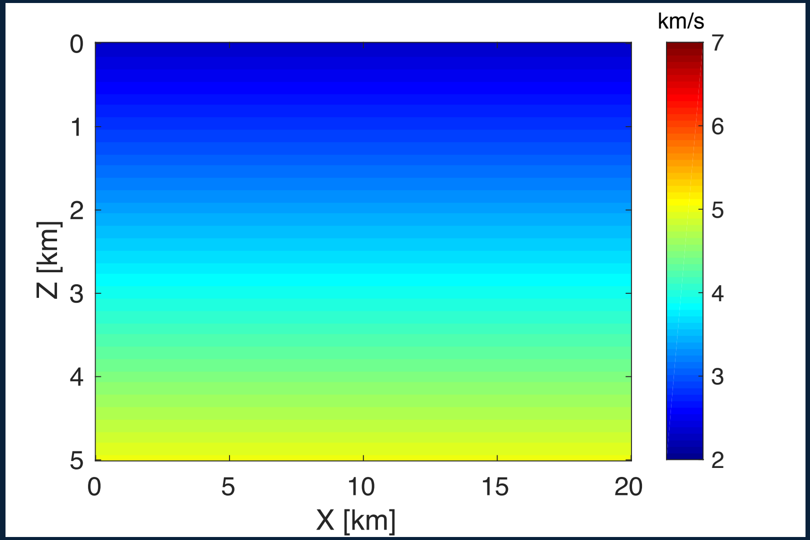
Overthrust model

Model size: 5 km x 20 km
Frequency bands: 2-3Hz, 5-7Hz, 7-9Hz
Number of shots: 50
Number of receivers: 50
Choices of λ : 1e-2, 1e0, 1e2
Choices of γ : 1e-1, 1e-3, 1e-5
Grid spacing: 25m

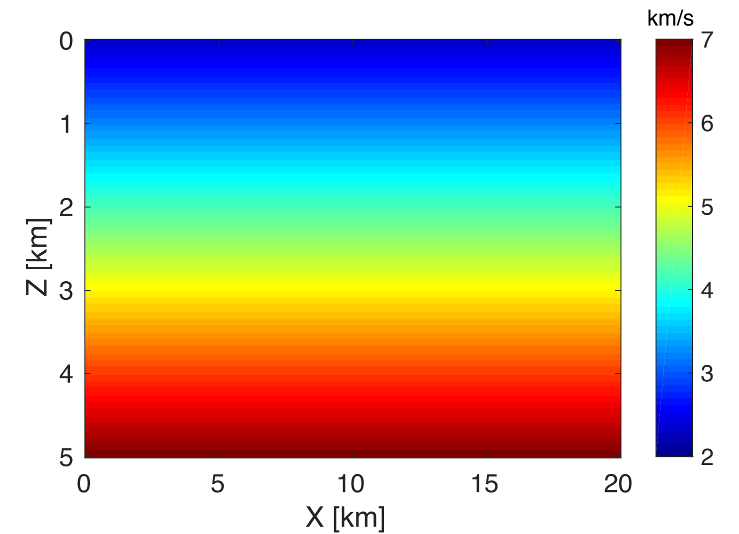


Numerical experiment

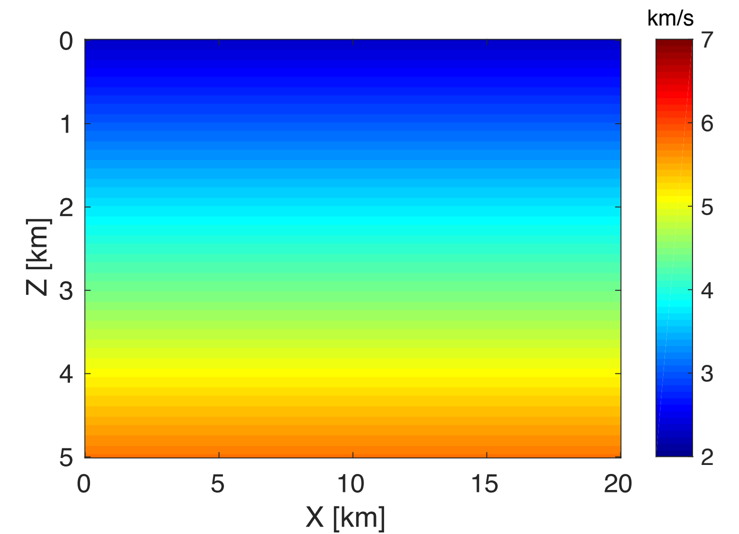
Initial models $\theta = \frac{\pi}{4}$



(a) Initial model $\mathrm{m}_1^{(0)}$



(b) Initial model $\mathrm{m}_2^{(0)}$

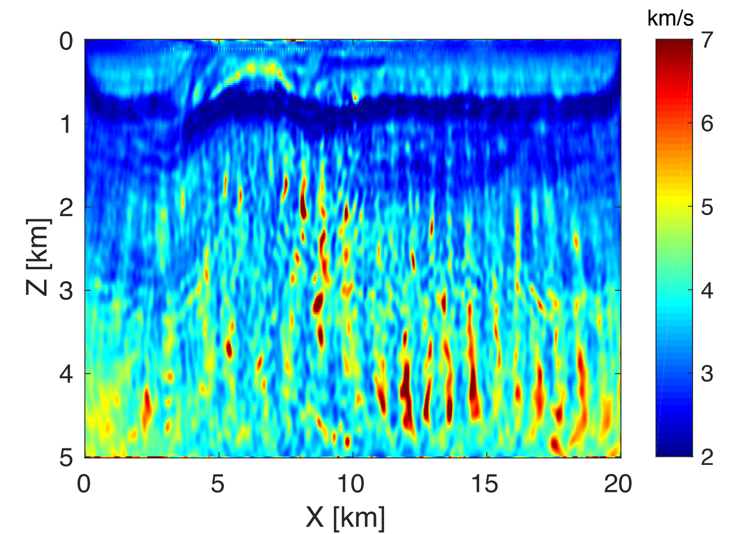


(c) Initial model

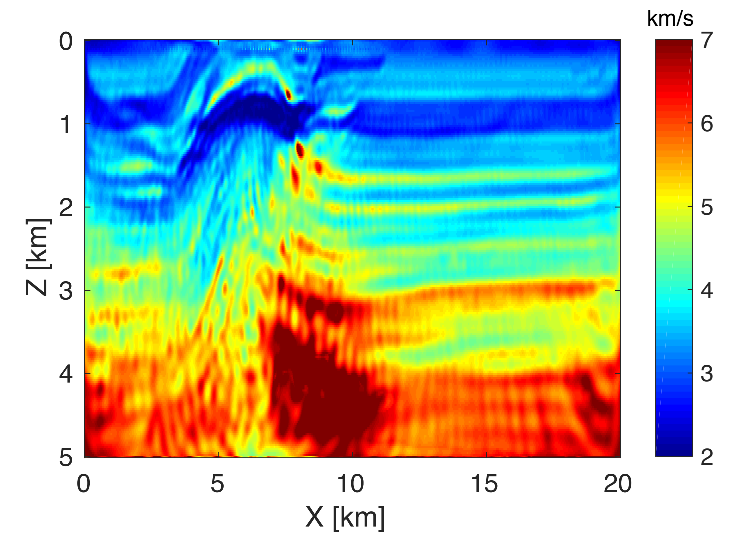
$$\mathbf{m}_3^{(0)} = \sin^2\theta^{(0)}\mathbf{m}_1^{(0)} + \cos^2\theta^{(0)}\mathbf{m}_2^{(0)}$$

Numerical experiment

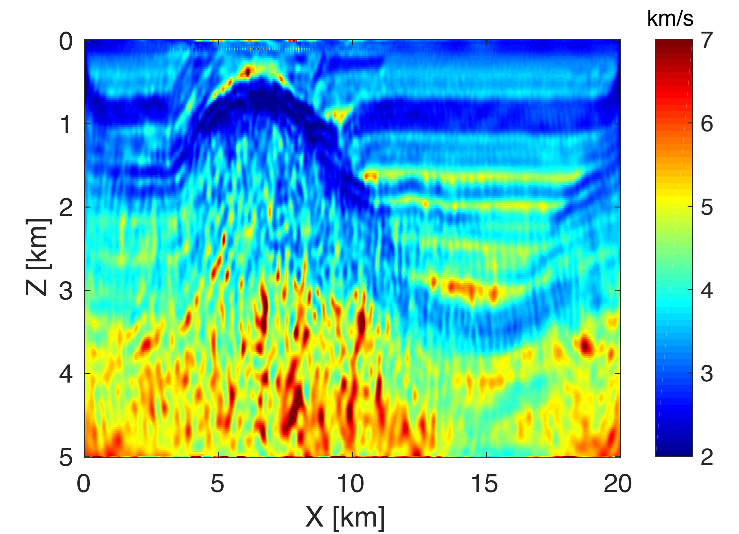
Results of FWI



(a) Result of $\mathrm{m}_1^{(0)}$



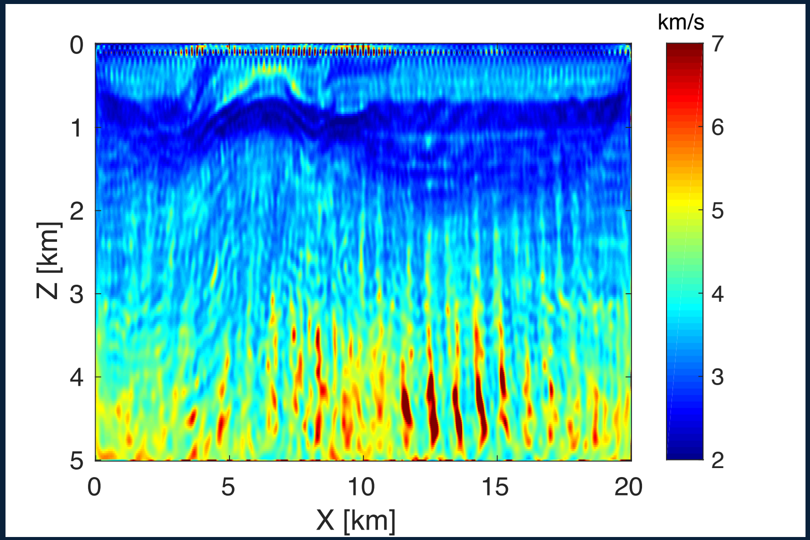
(b) Result of $\mathrm{m}_2^{(0)}$



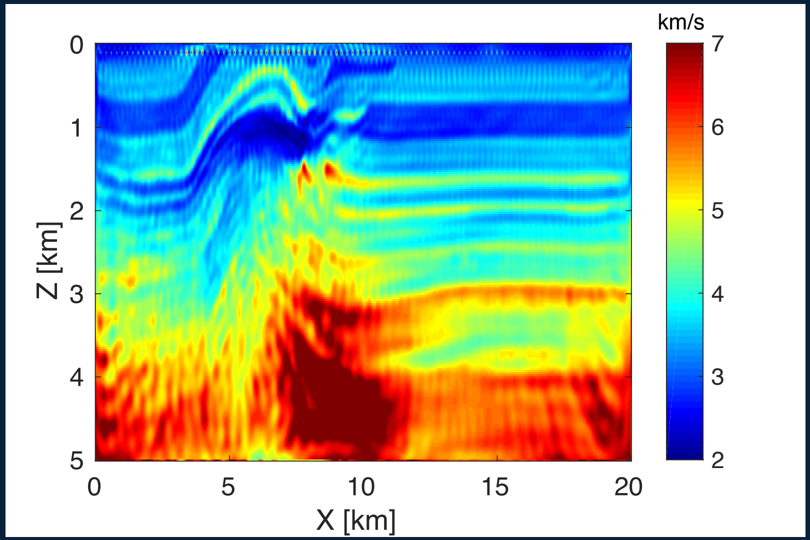
(c) Result of $\mathrm{m}_3^{(0)}$

Numerical experiment

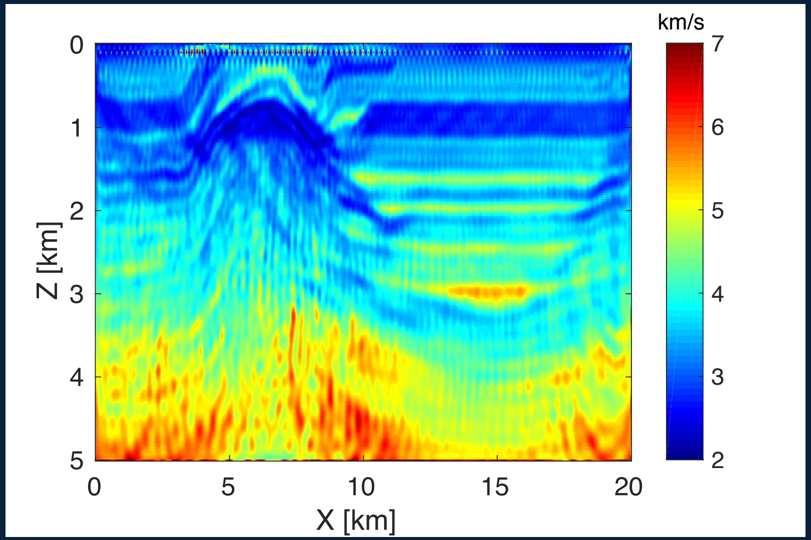
Results of WRI



(a) Result of $\mathrm{m}_1^{(0)}$



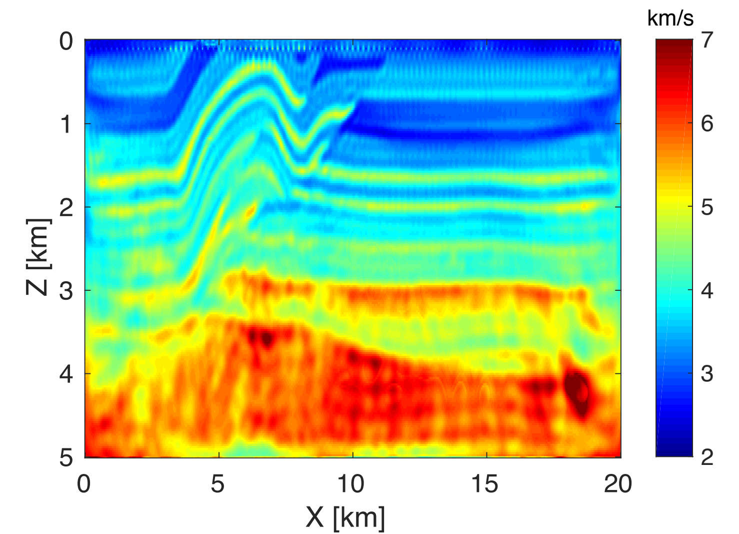
(b) Result of $\mathrm{m}_2^{(0)}$



(c) Result of $\mathrm{m}_3^{(0)}$

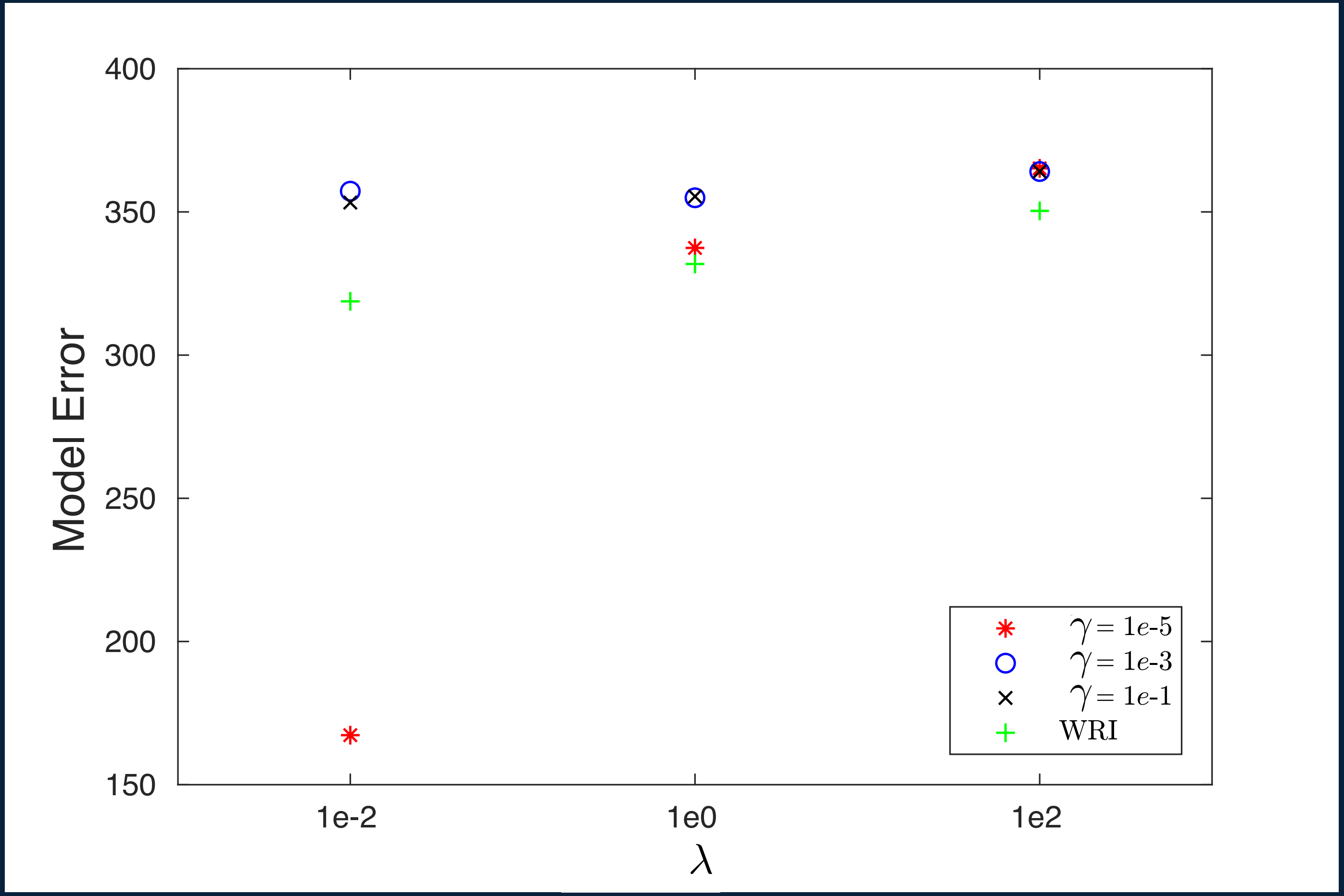
Numerical experiment

Results of LRWI



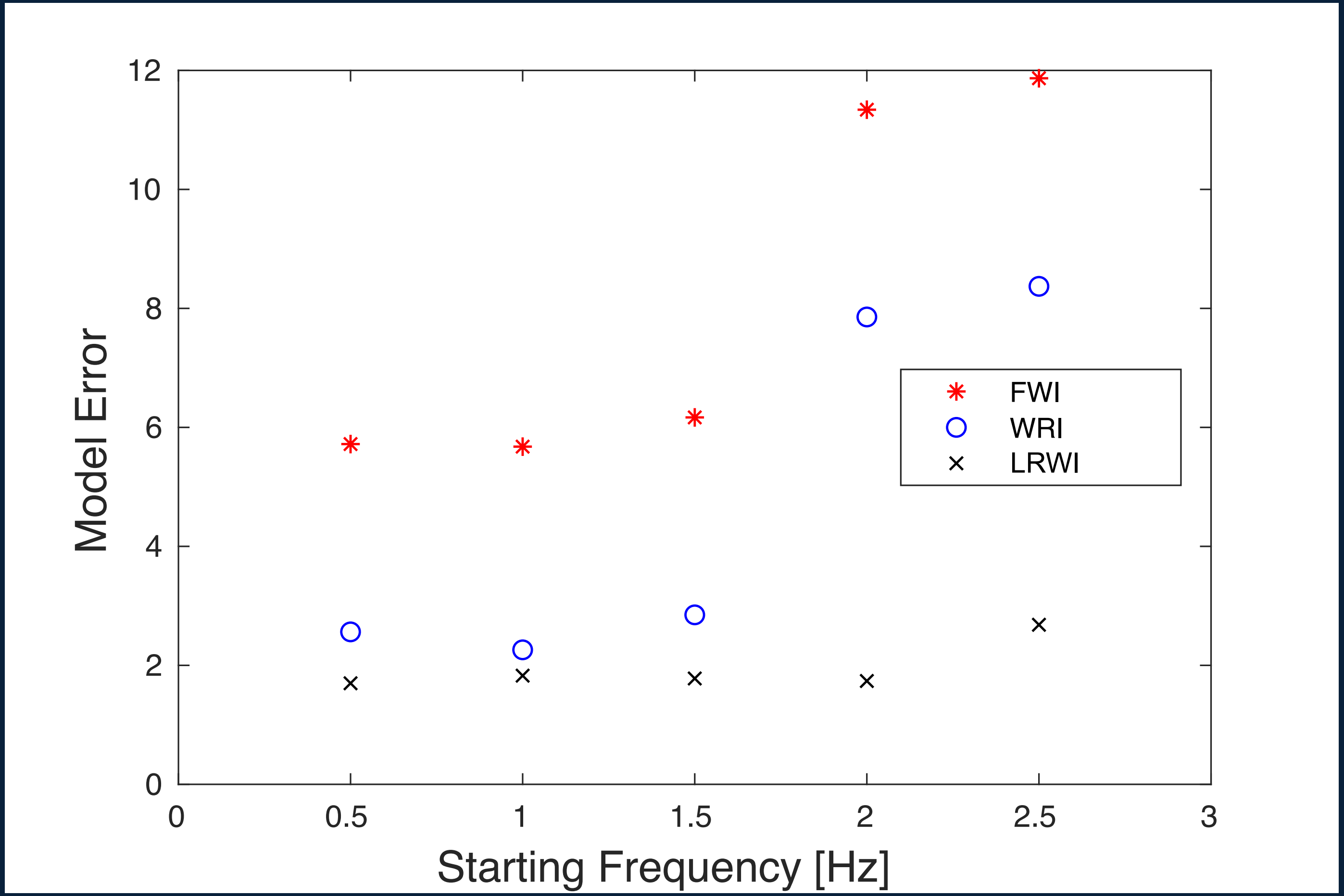
Numerical experiment

Influence of the penalty parameters



Numerical experiment

Comparisons of capability w.r.t the lowest frequency



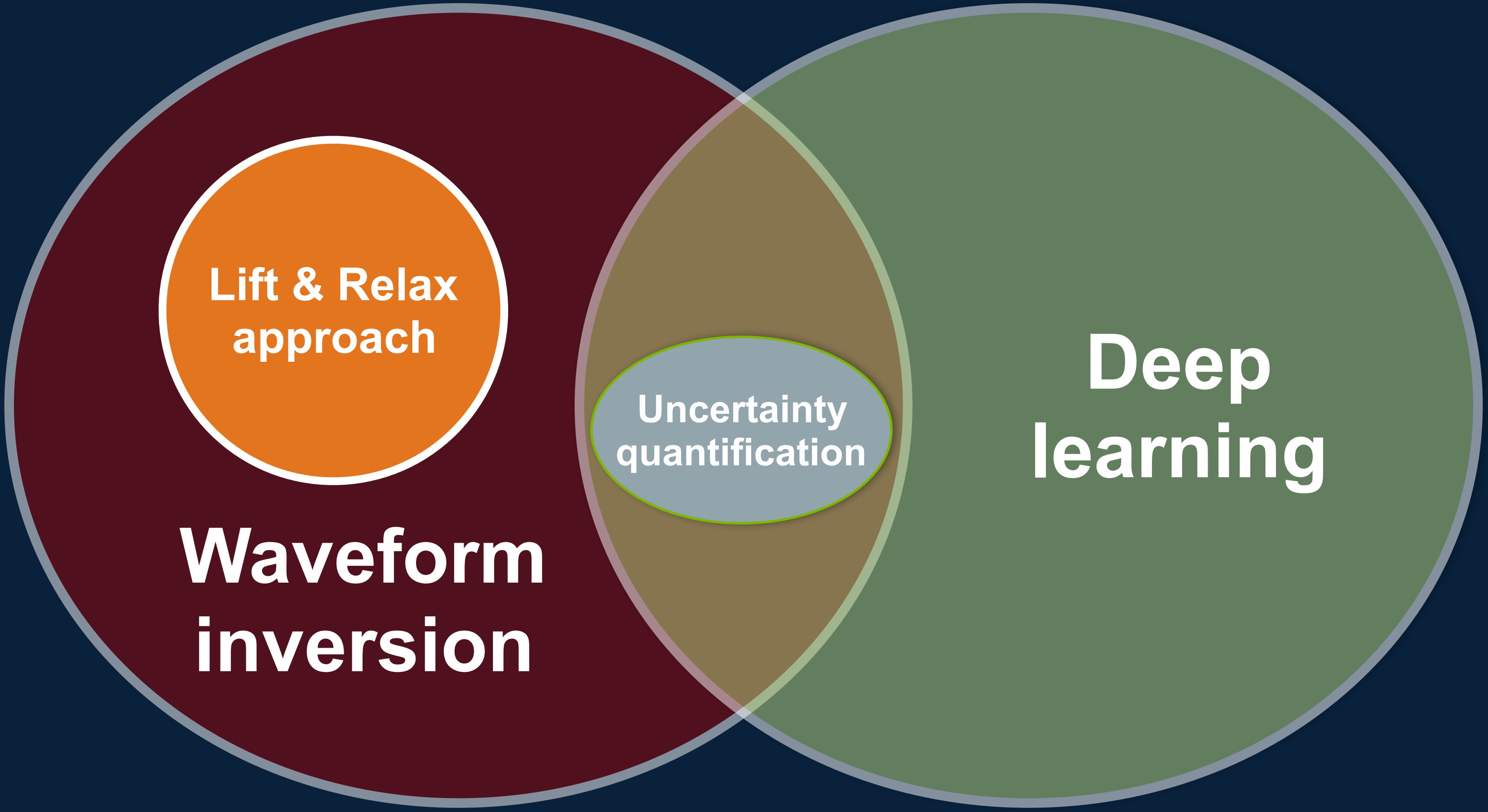
Conclusions

1. Through lifting the unknown parameters and relaxing the wave-equation constraint, we propose the LRWI method to mitigate the local minima that conventional FWI suffers from.

2. LRWI can start with poorer initial models and lower starting frequency compared to the conventional FWI and WRI.

3. The computational cost of LRWI is larger than the conventional FWI and WRI.

Waveform inversion and deep learning



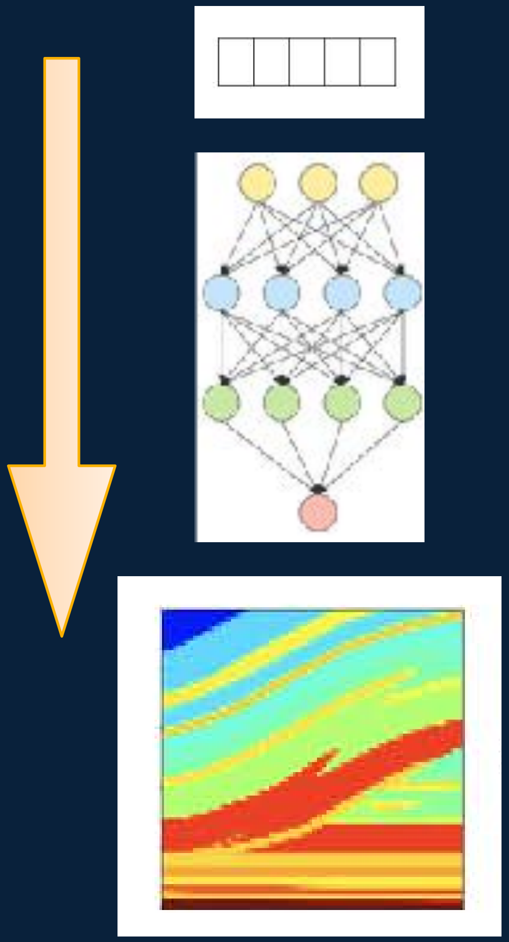
Lift & Relax approach

Waveform inversion

Uncertainty quantification

Deep learning

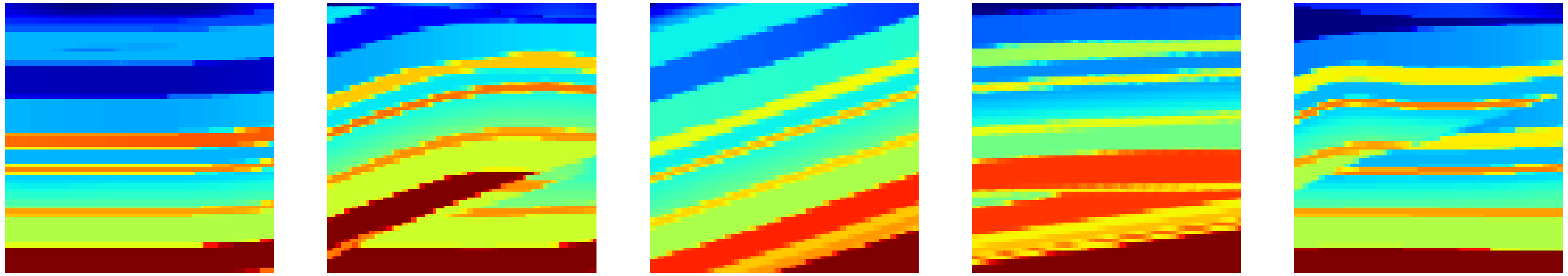
Deep prior generator



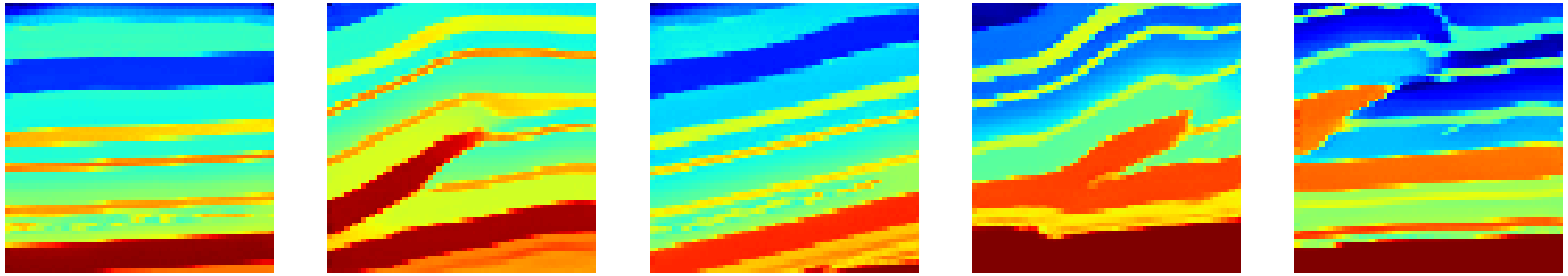
Input x

Output m

(a) Images from given models

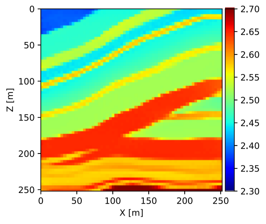


(b) Images generated by deep network

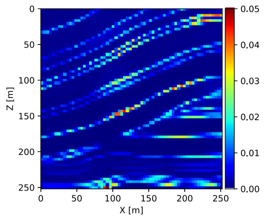


Research directions

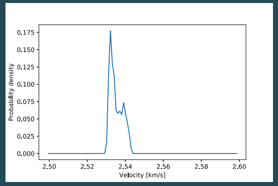
Applications to statistical seismic inversions



MAP

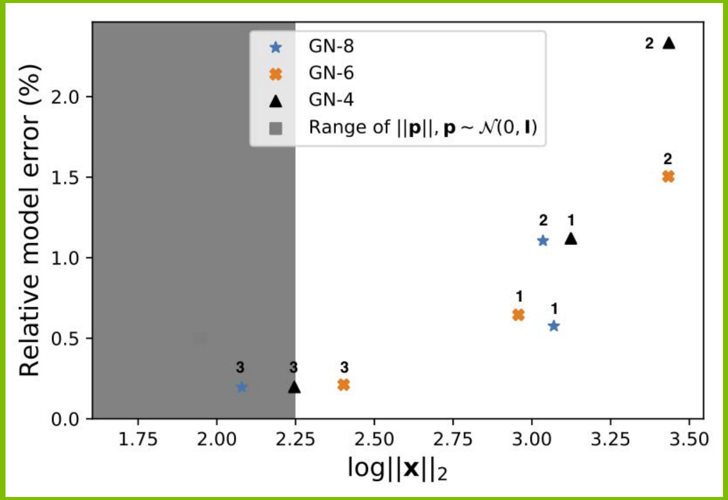


Standard deviation



Distribution

Quality control for the generative network



Visualizable and quantitative measure for the quality control of the deep generator