Uncertainty Quantification of Velocity Models and Seismic Imaging

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Seismic Inversion: Challenges

- Single image
- Sensitive to initial model (non-convex, non-linear problem)
- Expensive forward solves
  - (finite difference, finite element)
- No uncertainty quantification
  - How large is the trap?

Example From: http://pysit.readthedocs.org/en/release/0.5/quick_start/marmousi.html#details-of-parallelmarmousi-py
Bayesian Seismic Inversion

• Goal: Estimate distribution of velocity models given data.

• Challenges
  – Large dimensionality (128 X 384 pixels)
    • Displaying & Interpreting error
    • Calculating or storing covariance
  – Expensive forward solvers
  – Non-linear forward model
    • Multimodal & non-Gaussian model distribution

• Solution: Markov-Chain Monte Carlo & fast forward model
Outline

• Bayesian model Inversion
  – Bayesian inverse problem
  – MCMC Sampling

• Fast Forward Solver
  – Field Expansion Method
  – Gradient Adaptation

• Results
  – Simple synthetic
  – Marmousi based model
Bayesian Model Inversion

\[ d = f(m) + n \]

\[ n \sim N(0, \Sigma) \]  
Gaussian noise

\[ f(m) \]  
wave equation forward model

\[ d \]  
Observed data

Likelihood function

\[ L(m) \equiv p(d \mid m) \propto \exp \left[ -\frac{1}{2} (f(m) - d)^T \Sigma^{-1} (f(m) - d) \right] \]

Posterior Distribution

\[ p(m \mid d) = \frac{p(d \mid m) p(m)}{p(d)} \]
MCMC: Adaptive Metropolis-Hastings Summary

• Generates samples directly from posterior $(m_0, m_2, \ldots, m_n)$
  – Non parametric
  – ’Invariant’ to starting model
  – Self tuning proposal distribution

• Requirements & Drawbacks
  – 100,000 evaluations of Likelihood
  – Requires 200< parameters

• Finite difference, finite element
  – Too slow
  – Large dimensionality

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Fast Forward Solve: Field Expansion

- Reduced model space & fast
  - Discrete layers with perturbations (no pixels)
  - $O(LPN^2\log(P))$
  - $L$: number of layers $P$: spatial modes $N$: Taylor series terms

- Acoustic Helmholtz solver (Frequency domain)
  - Parallelizable over shot

- Limited velocity models
  - Piecewise constant
  - Continuous layers

Field Expansion: Gradient Adaptation

- Algorithm scales as \(O(LP^2\log(P))\)
  - Linear in \# of layers \(L\)
- Parameterize model to master layers with gradient
  - Low dimensionality & complex models
    - Datta and Sen 2016: global velocity model estimation
    - Fraser and Sen 1985: synthetic seismograms
    - Zelt and Smith 1992: Tomography
Field Expansion: Comparison

- Top: Marmousi
- Bottom: Foothills
- Field expansion: 5 master layers, 7 nodes per interface, 10 layers per gradient
- Pixel Model: 46,848 parameters, Field Expansion: 41 parameters

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Simple Model

- Velocity: (1500, 2500, 3000) m/s
- 5Hz, Single source, 256 receivers, Gaussian noise: SNR 1.2
- MCMC Run: 100,000 iterations
  - Migrate reflector for each velocity model
Simple Model: Realizations from MCMC

- MCMC Run: 100,000 iterations, discard first 50,000
- Showing every 100th sample
- Layers compensate thickness for velocity
Simple Model: Visualizing Error

- Uncertainty increases with depth
- Error bars (standard deviation) misrepresent uncertainty
  - Stable shape
  - Depth more uncertain
Simple Model: Quantities of Interest

- Relative height more stable than depth (165 +/- 15m, 2200 +/- 135m)
- Focusing on quantities of interests simplifies analysis
  - Reduces dimensionality
  - Easy to interpret
  - Visualize non-Gaussian easily
Marmousi Example

- 3Hz single shot, 256 receivers, SNR 0.75, 41 velocity model parameters
- MCMC Run: 100,000 iterations, discard first 50,000

Focus on stability of images (reflectors of interest)
Posterior Distribution: Velocity Models

- 50,000 models of 100,000 discarded, 6 models randomly selected
- Shallow velocity structure better constrained than deeper
- Too complex to show error bars
Posterior Distribution: Migrated Images

- Generated reflectivity model $\rightarrow$ travel times
- Zero offset migrate travel times with each velocity model

- Upper structure more stable than lower
- Red anticline shape poorly constrained
- Salt flank/sill a trap? (D,F)

1 km
Posterior Distribution: Quantities of Interest

- Ran MCMC 50 times → 2.5 million posterior samples
- Deeper red anticline area, poorly constrained
  - Non-Gaussian distribution
  - Significant samples near zero area
Summary

• Presented framework for uncertainty quantification
  – Reduced dimensionality of forward solver
  – Adapted field expansion to model gradients
  – Combined field expansion with adaptive metropolis hasting algorithm

• Demonstrated algorithm on synthetic models
  – Simple & Marmousi velocity models
  – Showed methods to visualize error
  – Focused on quantities of interest

• Robustness of algorithm
  – Estimates of uncertainty from single MCMC run
  – Handles incorrect parameterization
  – Not truly invariant to starting models, tolerates 20% gradient perturbations
Initial Model Choice: Linear Gradient

- Re-ran MCMC inversion with linear gradient initial model 50 times
  - Linear Gradient: 1500 m/s – 4300 m/s
- Distributions for quantities of interest within 1 STD
  - Cross sectional area matches well
Initial Model Choice: Incorrect Linear Gradient

- Ran 20 chains for 8 linear gradients
  - Slowest: 1500-2900 m/s  Fastest: 1500-5400 m/s
- Poor convergence with 25% gradient perturbation
  - MCMC didn’t find global minima
  - Quantities of interest biased
- Below 20% perturbation consistent quantities of interest
Initial Model Choice: Incorrect Parameterization

- 3 – 5 layers, 3 – 11 nodes
- 10 trials per layer node combination

- Number of nodes significant impact.
- Number of layers lesser impact.
Convergence Results

- Dashed lines standard deviation from all 2.5 million samples (50 runs)
- Good estimate of area standard deviation achieved within 50,000 sample (1 run)
  - Both true and initial starting models
Adaptive Metropolis-Hastings: Outline

• Select initial model $m_0$
• Evaluate $L_0 = L(m_0) \exp \left[ -\frac{1}{2} (f(m) - d)^T \Sigma^{-1} (f(m) - d) \right]

For $i = 1:100,000$

- $m_* = m_{i-1} + n$, $n \sim N(0, C_i)$
  
  $$C_i = S_d (\text{Cov}[m_0, \ldots, m_{i-1}] + \varepsilon I_d)$$

- With probability $L(m_*)/L(m_{i-1})$
  
  • $m_i = m_*$

- Else
  
  • $m_i = m_{i-1}$

Outline

• Motivation

• Bayesian model Inversion
  – Bayesian inverse problem
  – MCMC Sampling

• Field expansion method

• Results
  – Simple synthetic
  – Marmousi based model

• Validation
  – Inaccurate starting models
  – Convergence
Seismic Inversion: Overview

- Marmousi acoustic velocity model
- 6 finite difference shots
Outline

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Seismic Inversion: Overview

- Record gathers
- Estimate velocity model
Seismic Inversion: Overview

- Need accurate initial model
- Local optimization
Extra Slides
Field Expansion: Wood’s Anomalies

- Diffraction grating
- Resonant mode
- Energy trapped in top layer
- Cannot be removed by large domain size

- **Solution: Absorption**

Removing Wood’s Anomalies

- 3 layer velocity model (1500, 2500, 3500 m/s)
- **Solution**: top layer absorptive
Wood’s Anomalies: Convexity & Cost Function

- 3 layer velocity model (Top Layer, depth: 500 m, Velocity 1100 m/s)
- Anomalies create local minima
Comparison to Existing Methods

- 3 layer model sinusoidal perturbation
- Seismic Unix finite difference solver
Comparison to Existing Methods: Frequency domain

- 3 Flat layer model, 3Hz (-3000, 3000 m)
- Pysit 4th order finite difference solver
Field Expansion: Finite Domain

- Finite domain size
  - Energy bleeds across domains

- Solution: Increase domain size
Bayesian Model Inversion

\[ d = f(m) + n \]

\( f(m) \): forward model

\( n \sim N(0, \Sigma) \): Gaussian Noise

Observed data

Likelihood function

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Fast Forward Solve: Field Expansion

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  - Reduced Model space & Fast
- Acoustic Helmholtz solver (Frequency domain)
  - Parallelizable over shot
- **Diffraction theory: Spatial frequency domain & repeating domain size**
Field Expansion: Algorithm

Algorithm 1 Perturbed Helmholtz Solver

Determine number of spatial modes $P$
Determine order of Taylor series $N$

for $p = 1$ to $P$ do

Construct source vector $\vec{R}_p$
Construct velocity model matrix $A_p$

Solve $A_p \vec{u}_{p,0} = \vec{r}_{p,0}$ to get $[u_{p,m,0}, d_{p,m,0}]$

for $n = 1$ to $N$ do

from $d_{n-1}$ and $u_{n-1}$ Construct $\vec{R}_{p,n}$

Solve $A_p \vec{z}_{p,n} = \vec{r}_{p,n}$ to get $[u_{p,m,n}, d_{p,m,n}]$

end for

end for

Use $[u_{p,m,n}, d_{p,m,n}]$ to calculate field $v_m$ with Equation 9

\[
v_m(x, y) = \sum_{p=-\infty}^{\infty} d_{p,m} e^{i(\alpha_p - \beta_p m (y-a_m))} + u_{p,m} e^{i(\alpha_p + \beta_p m (y-a_m))}
\]

• $O(LP N^2 \text{Log}(P))$
Field Expansion: Flat Layered

Helmholtz Equation

$$\nabla^2 v_m + k_m v_m = 0 \quad k_m = \frac{2\pi f}{c_m}$$

Expression for field

$$v_m(x, y) = \sum_{p=-\infty}^{\infty} d_{pm} e^{i(\alpha_p - \beta_{pm}(y-a_m))} + u_{pm} e^{i(\alpha_p + \beta_{pm}(y-a_m))}$$

Boundary Conditions

$$v_{m-1} - v_m = \xi_m \quad \text{at} \quad y = a_m + g_m(x)$$

$$\partial_N v_{m-1} - \partial_N v_m = \psi_m \quad \text{at} \quad y = a_m + g_m(x).$$

System of equations

$$A_p \vec{z}_p = \vec{r}_p,$$
Field Expansion: Flat Layered

- \( A_p \) penta-diagonal, \( O(L) \)
- Solve for each \( p \), \( O(PL) \)
- Plug back into field expression \( O(PL \ Log(P)) \)

System of equations

\[
A_p \vec{z}_p = \vec{r}_p,
\]

\[
\vec{z}_p = (u_0, d_1, u_1, \ldots, d_{M-1}, u_{M-1}, d_M)^T
\]

Apply Point Source

\[
\vec{r}_p = (\xi, \psi, \ldots, 0, 0)^T
\]

\[
\xi_p(x, y) = \frac{1}{2id} e^{i(\alpha_p(x-x_0) + \beta_p|y-y_0|)}
\]

\[
\psi_p = \partial_{N_m} \xi_p.
\]
Field Expansion: Perturbed Layers

- Solve for $N = 0 \rightarrow$ flat layer case
- Solve $N = 1, 2, 3, 4$, etc.

**Flat Field**

$$v_m(x, y) = \sum_{p=-\infty}^{\infty} \sum_{n=0}^{\infty} d_{pm,n} e^{i(\alpha_p - \beta_p m(y - \bar{a}_m))} + u_{pm,n} e^{i(\alpha_p + \beta_p m(y - \bar{a}_m))}$$

**Perturbed Field**

$$v_m(x, y) = \sum_{p=-\infty}^{\infty} d_{pm,n} e^{i(\alpha_p - \beta_p m(y - \bar{a}_m))} + u_{pm,n} e^{i(\alpha_p + \beta_p m(y - \bar{a}_m))} z^n$$

**Recursion Relation**

$$A_p \vec{z}(p,n) = \vec{R}(p,n)$$

$$\vec{z}(p,n-1) \rightarrow \vec{z}(p,n)$$
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